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Non-commutative principal chiral models

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Abstract: Twisted Eguchi-Kawai reduced chiral models are shown to be formally equivalent to a U(1) non-commutative parent theory. The non-commutative theory describes the vacuum dynamics of the non-commutative charged tachyonic field of a brane system. To make contact with the continuum non-commutative theory, a double scaling large N limit for the reduced model is required. We show a possible limiting procedure, which we propose to investigate numerically. Our numerical results show substantial consistency with the outlined procedure.

Keywords: Lattice Quantum Field Theory, 1/N Expansion, p-branes, Non-Commutative Geometry.

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Contents

1. Introduction

Twisted Eguchi-Kawai (TEK) reduced models [1] provide a non-perturbative definition of certain non-commutative field theories [2, 3, 4]. It has been shown that the U(N) gauge theory with gauge fields obeying twisted boundary conditions over the non-commutative torus $\mathbb{T}^D_\Theta$ is equivalent to a U($\tilde{N}$) gauge theory, with $\tilde{N}$ suitably chosen, over the non-commutative torus $\mathbb{T}^D_{\Theta'}$ with gauge fields obeying periodic boundary conditions [4]. This is a consequence of a more general fact, known as *Morita equivalence*.

We would like to propose an application of the formalism of ref. [4, 5] to principal chiral models, possibly providing numerical evidence, in order to show that two-dimensional TEK reduced chiral models can be considered as a non-perturbative definition of a non-commutative field theory.

Explicitly, we are going to show the equivalence of TEK principal chiral model with symmetry group U(N) to a non-commutative U(1) lattice theory compactified on a torus with periodic boundary conditions. We will describe the corresponding non-commutative theory, i.e. its action and symmetries, coupling constant and dimensionful non-commutativity parameter. We will eventually try to define a procedure which can lead to a sensible continuum limit, and therefore define the set of values of the coupling constants and of the symmetry parameter N of the original TEK theory relevant for a numerical check of the consistency of the whole approach.

The reduced model, in this context, should be considered in a different limit from the original planar one. In the planar limit one has to send $N \to \infty$ while keeping the lattice coupling constant β fixed. The continuum limit is then reached as one sends $\beta \to \infty$, because the RG relation $a \propto e^{-c\cdot\beta}$ exists between the lattice spacing a and the lattice coupling constant β. The limit is called planar since the large N dominant Feynman graphs are, in this limit, planar, i.e. they can be drawn on a Riemann surface of genus $g = 0$.

For the TEK model to reproduce the corresponding non-commutative field theory we must consider a different limiting procedure to approach the continuum limit. This procedure is called *double scaling limit*, since β and N must be sent to infinity in a correlated manner. We know that in the non reduced original field theory this corresponds to taking into consideration also non-planar graphs, analogously to what happens in matrix models of $2D$ gravity: contributions from higher genus topologies imply a higher symmetry in the problem, which enables to make contact with the theory one wants to reproduce. Quite the same thing happens in non-commutative field theory, since the interaction vertex is invariant only up to cyclic permutations of the momenta, and therefore one needs to keep track of the cyclic order in which lines emanate from vertices in a given Feynman diagram. Non-commutative Feynman graphs are thus ribbon graphs drawn on Riemann surfaces of particular genus, in complete analogy with what happens in the ordinary large N limit of field theories. Non-planar contributions thus naturally arise in the present context of non-commutative field theories.

Principal chiral models in $D = 2$ are since long [6] known to be in a sense[1] a simpler counterpart of lattice Yang-Mills theory in $D = 4$. Also as far as the reduced TEK models of the two theories are concerned, the case of chiral models is more tractable, and even numerical simulations are easier and more conclusive. For this reason we consider principal chiral model as an important test for the non-commutative interpretation of TEK reduced models.

Furthermore, the non-commutative theory arising from the TEK reduced principal chiral model turns out to possess exactly the action needed to describe the vacuum dynamics of the non-commutative charged tachion of a certain brane system [7].

We begin with a general description of non-commutative field theories, in order to fix the notation, and describe the particular non-commutative field theory we are interested in, i.e. $2D$ principal chiral models. We then pass to a short analysis of the reduced models and of the consequences of our numerical results. In the final section we demonstrate the equivalence between TEK reduced chiral models and non-commutative U(1) principal chiral models on the lattice, indicating the procedure needed to approach a sensible continuum limit. This will eventually lead to a concrete proposal of a numerical study of the theory reread in this new light. The physical interpretation of the presented non-commutative theory is then briefly sketched. We include the results of our Monte Carlo simulations, which directly enforce the correctness of the whole procedure.

2. Non-commutative field theories

We will briefly recall the so called Weyl quantization procedure for field theories on non-commutative spaces. Let us consider for simplicity a scalar field theory on euclidean $\mathbb{R}^D$,

[1] This correspondence is exact in $D = 1$ and $D = 2$ respectively.

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defined by some action $S[\phi]$, and whose partition function is as usual

$$Z = \int \mathcal{D}\phi \; e^{-S[\phi]} \,. \tag{2.1}$$

In order to pass from ordinary to non-commutative space-time, we replace the local coordinates x_μ by hermitean operators $\hat{x}_\mu$ which have the following commutations relations:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} \,, \tag{2.2}$$

where $\theta_{\mu\nu} = -\theta_{\nu\mu}$ is a real valued anti-symmetric matrix with the dimensions of length squared. Implementing the substitution $x_\mu \to \hat{x}_\mu$ we obtain the *Weyl operator* $\hat{\phi}$ corresponding to the field ϕ.

One can define an operator which transforms a field into its Weyl operator:

$$\begin{aligned} \hat{\phi} &= \int \mathrm{d}^D x \phi(x) \hat{\Delta}(x) \,, \\ \hat{\Delta}(x) &= \int \frac{\mathrm{d}^D k}{(2\pi)^D} e^{ik_\mu \hat{x}^\mu} e^{-ik_\mu x^\mu} \,. \end{aligned} \tag{2.3}$$

It is possible to introduce an anti-hermitean derivation $\hat{\partial}_\mu$ through the commutations relations

$$[\hat{\partial}_\mu, \hat{x}_\nu] = \delta_{\mu\nu} \,, \qquad [\hat{\partial}_\mu, \hat{\partial}_\nu] = 0 \,. \tag{2.4}$$

The operator $\hat{\partial}_\mu$ obviously satisfies the following relations:

$$\begin{aligned} [\hat{\partial}_\mu, \hat{\phi}] &= \int \mathrm{d}^D x \; \partial_\mu \phi(x) \hat{\Delta}(x) \,, \\ [\hat{\partial}_\mu, \hat{\Delta}(x)] &= \hat{\partial}_\mu \hat{\Delta}(x) \,. \end{aligned} \tag{2.5}$$

This implies that, given the generator of the translations $e^{v_\mu \hat{\partial}^\mu}$, $v_\mu \in \mathbb{R}$, which satisfies

$$e^{v_\mu \hat{\partial}^\mu} \hat{\Delta}(x) e^{-v_\mu \hat{\partial}^m u} = \hat{\Delta}(x+v) \,, \tag{2.6}$$

the operation $\mathrm{Tr}\,\hat{\Delta}(x)$ is independent of x for any trace on the algebra of operators. Therefore the integration of the fields on the space-time is represented by

$$\mathrm{Tr}\,\hat{\phi} = \int \mathrm{d}^D x \; \phi(x) \,, \qquad \mathrm{Tr}\,\hat{\Delta}(x) = 1 \,. \tag{2.7}$$

This enables to define the inverse of the correspondence between fields and operators:

$$\phi(x) = \mathrm{Tr}\left(\hat{\phi}\hat{\Delta}(x)\right) . \tag{2.8}$$

In conclusion, in order to pass from ordinary to non-commutative field theory, we need to implement the following substitutions:

$$\phi \to \hat{\phi} \,, \qquad \partial_\mu \phi \to [\hat{\partial}_\mu, \hat{\phi}] \,. \tag{2.9}$$

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The substantial difference between the two theories comes from the definition of products of fields. In the non-commutative case one has, for $\hat{\phi}_3 = \hat{\phi}_1 \hat{\phi}_2$

$$\begin{aligned}
\phi_3(x) &= \mathrm{Tr}\left(\hat{\phi}_1 \hat{\phi}_2 \hat{\Delta}(x)\right) \\
&= \frac{1}{\pi^D |\det\theta|} \int\int \mathrm{d}^D\, y \mathrm{d}^D z \phi_1(y)\phi_2(z) e^{-2i(\theta^{-1})_{\mu\nu}(x-y)_\mu (x-z)_\nu} \\
&= \phi_1(x) \exp\left(\frac{i}{2}\overleftarrow{\partial_\mu}\theta_{\mu\nu}\overrightarrow{\partial_\nu}\right)\phi_2(x) \equiv \phi_1(x) \star \phi_2(x) \\
&= \phi_1(x)\phi_2(x) + \sum_{n=1}^{\infty}\left(\frac{i}{2}\right)^n \frac{1}{n!}\theta^{i_1 j_1}\cdots\theta^{i_n j_n}\partial_{i_1}\cdots\partial_{i_n}\phi_1(x)\partial_{j_1}\cdots\partial_{j_n}\phi_2(x)\,, \qquad (2.10)
\end{aligned}$$

which defines the *star* or *Moyal product* of the fields.

2.1 Non-commutative U(N) principal chiral models

The theory we would like to study is defined in a D dimensional euclidean space by the following action, multiplied by a suitable coupling constant

$$S = \beta N \int \mathrm{d}^D x\ \mathrm{tr}_{(N)}\left(\partial_\mu U(x) \partial^\mu U^\dagger(x)\right), \qquad (2.11)$$

where the sum over $\mu = 1, 2, \ldots, D$, and a flat euclidean metric are intended.[2] The unitary matrices $U_{ij}(x)$, with $i, j = 1, 2, \ldots, N$ satisfy the condition

$$\left(U^\dagger(x) U(x)\right)_{ij} = \delta_{ij} \qquad \forall\, x\,. \qquad (2.12)$$

The partition function is defined over the usual Haar measure

$$Z = \int \mathrm{d}U\ e^{-S[U]}\,. \qquad (2.13)$$

The action is naturally invariant under the transformations (where $V_L, V_R \in \mathrm{U}(N)$)

$$U \to V_L U, \qquad U \to U V_R\,. \qquad (2.14)$$

In the present formalism, one needs to define the Weyl operators as

$$\hat{U} \equiv \int \mathrm{d}^D x\ \hat{\Delta}(x) \cdot U(x)\,, \qquad (2.15)$$

where the operator $\hat{\Delta}(x)$ is the one defined in (2.3). The action is rewritten as

$$S = \beta N\, \mathrm{Tr}\left\{\mathrm{tr}_{(N)}\left(\left[\hat{\partial}_\mu, \hat{U}\right]\left[\hat{\partial}^\mu, \hat{U}^\dagger\right]\right)\right\}\,, \qquad (2.16)$$

where Tr is the trace operator over coordinates, while $\mathrm{tr}_{(\mathrm{N})}$ is the (finite-dimensional) trace in the fundamental representation of the U(N) group.

[2]In what follows we will restrict to the case $D = 2$.

The corresponding non-commutative fields $\mathcal{U}(x)$, defined through the inverse transformation (2.8), satisfy the star-unitarity condition[3]

$$\mathcal{U}(x) \ \star \ \mathcal{U}(x)^\dagger = \mathcal{U}(x)^\dagger \ \star \ \mathcal{U}(x) = \mathbb{I}_N \, . \tag{2.17}$$

The action is given by the obvious translation of eq. (2.16)

$$S = \beta N \int \mathrm{d}^D x \ \mathrm{tr}_{(N)} \left(\partial_\mu \mathcal{U}(x) \star \partial^\mu \mathcal{U}(x)^\dagger \right) . \tag{2.18}$$

The invariance of the action naturally reads, owing to cyclicity of the trace $\mathrm{tr}_{(N)}$,

$$\mathcal{U}(x) \to g_L \, \mathcal{U}(x) \qquad \mathcal{U}(x) \to \mathcal{U}(x) \ g_R \, , \tag{2.19}$$

where the $N{\times}N$ matrices g_L, g_R are ordinary unitary matrices, therefore satisfying $gg^\dagger = \mathbb{I}_N$.

3. The reduced model

The principal chiral model, in $D = 2$, is defined on the lattice via the usual substitution of the derivative with a finite difference

$$\partial_\mu \, \mathrm{U}(x) \ \to \ \frac{U_{x+a\hat{\mu}} - U_x}{a} \, , \tag{3.1}$$

and the resulting action reads

$$S = -\beta N \sum_x \sum_{\mu=1,2} \mathrm{tr}_{(N)} \left[U_x U^\dagger_{x+\mu} + U_{x+\mu} U^\dagger_x \right] . \tag{3.2}$$

The naive Eguchi-Kawai reduction prescription $U_x \to U$ is clearly not applicable in this context. Instead, one can resort to the TEK prescription, which is defined as

$$U_{(x_1,x_2)} \ \to \ \Gamma_1^{x_1} \Gamma_2^{x_2} \, \mathrm{U} (\Gamma_2^\dagger)^{x_2} (\Gamma_1^\dagger)^{x_1} \, , \tag{3.3}$$

where the *twist matrices* Γ_μ obey the Weyl-'t Hooft algebra

$$\Gamma_\mu \Gamma_\nu = \exp \left[\frac{2\pi i}{N} n_{\mu\nu} \right] \Gamma_\nu \Gamma_\mu \, , \tag{3.4}$$

where N is the parameter of the symmetry group $\mathrm{U}(N)$ or $\mathrm{SU}(N)$ (and thus of the matrix U) and $n_{\mu\nu}$ is an integer valued antisymmetric tensor, whose generic form in $D = 2$ is of course

$$n_{\mu\nu} \equiv \begin{pmatrix} 0 & M \\ -M & 0 \end{pmatrix}, \qquad M \in \mathbb{Z} \, . \tag{3.5}$$

For a given N and M the solution to (3.4) is provided, up to global $\mathrm{SU}(N)$ transformations, by the $N \times N$ shift and clock matrices[4]

$$\begin{aligned} S^{(M)}_{i,j} &\equiv \delta_{i+M,j} \, , \\ C_{i,j} &\equiv e^{(2\pi i/N)(i-1)} \delta_{i,j} \, . \end{aligned} \tag{3.6}$$

[3] since $\det(f \star g) \neq \det(f) \star \det(g)$ we cannot take $\mathrm{SU}(N)$ as the symmetry group of the theory.

[4] The two matrices are a natural extension of the known 't Hooft twist matrices [8].

The two matrices Γ_μ will be given respectively by S and C.

Applying the reduction prescription to the action (3.2) gives

$$S_{\text{TEK}} = -\beta N \sum_{\mu=1,2} \operatorname{Tr}\left[U\Gamma_\mu U^\dagger \Gamma_\mu^\dagger + \text{h.c.}\right]. \tag{3.7}$$

We notice that the model shows two symmetries, namely:

1. $U \;\rightarrow\; \Gamma_1^{x_1}\Gamma_2^{x_2}U(\Gamma_2^{x_2})^\dagger(\Gamma_1^{x_1})^\dagger$;
2. $U \;\rightarrow\; z \cdot U, \qquad z \in \mathbb{Z}_N$.

The first symmetry is reminiscent of the space-time translational symmetry of the original model (indeed from (3.3) it is clear that the role of the Γ_μ is that of generators of translations in the dual lattice of the reduced theory) while the second represents the residual global symmetry $\mathrm{SU}(N) \times \mathrm{SU}(N)$ of the parent theory reduced to the center $\mathbb{Z}_N$ of the algebra of $\mathrm{SU}(N)$. In the case of a symmetry $\mathrm{U}(N) \times \mathrm{U}(N)$ the second symmetry is of course a $\mathrm{U}(1)$ symmetry.

The introduction of the TEK reduced model was originally motivated by its supposed equivalence with the parent theory (defined by the action of eq. (3.2)) in the large N limit. This equivalence should follow basically from two facts:

- the Schwinger-Dyson (SD) equations of the reduced theory and that of the parent theory are exactly the same in the large N limit, up to terms which are not invariant under the $\mathbb{Z}_N$ symmetry;
- if in all regimes the symmetry is not spontaneously broken, the two models possess exactly the same SD set of equations, and given the same initial conditions should coincide.

It has been found [9] that the invoked equivalence holds for the strong coupling region, while in the weak coupling the reduced theory is manifestly different from its parent version. Thus, although the second symmetry does not seem to be spontaneously broken, the SD based argument for the equivalence is not sufficient, at least in a certain regime, to completely identify the theory.

It therefore makes sense to ask whether the reduced principal chiral model can provide a non perturbative definition of *another* theory. Our claim is that this theory can be interpreted as the *non-commutative* $\mathrm{U}(1)$ *version of* (3.2), and that, in a suitable limit to be defined, it reproduces its continuum limit, i.e. the one specified in the Weyl operator notation by eq. (2.16) and in the non-commutative field notation by eq. (2.18).

4. Reduced chiral models and non-commutative chiral fields

We will show in what follows how to map the TEK model into the non-commutative version of the theory defined by the action (3.2). Given an integer valued vector $\mathbf{k} = (k_1, k_2)$, we introduce the $N \times N$ matrices

$$J_k = \Gamma_1^{k_1}\Gamma_2^{k_2} e^{\pi i (n_{12}) k_2 k_1 / N} = \Gamma_1^{k_1}\Gamma_2^{k_2} e^{\pi i M k_2 k_1 / N}. \tag{4.1}$$

The phase factor is given to symmetrically order the product of twist eaters. Incidentally, the J_k's have the same algebraic properties as the plane Weyl basis $e^{ik_i\hat{x}^i}$ for the continuum non-commutative field theory on the torus [5].

The relevant properties of these matrices are that there are only N^2 such matrices, owing to the periodicity properties

$$J_{N-k} = J_{-k} = J_k^\dagger \,, \tag{4.2}$$

and that they obey the orthonormality and completeness relations

$$\begin{aligned} \frac{1}{N}\mathrm{tr}_{(N)}\left(J_k J_q^\dagger\right) &= \delta_{k,q \pmod N} \,, \\ \frac{1}{N}\sum_{k\in\mathbb{Z}_N^2}(J_k)_{\mu\nu}(J_k)_{\lambda\rho} &= \delta_{\mu\rho}\delta_{\nu\lambda} \,. \end{aligned} \tag{4.3}$$

They therefore form a basis for the linear space $gl(N,\mathbb{C})$ of $N\times N$ complex matrices, and in particular one can expand a matrix U as

$$U = \frac{1}{N^2}\sum_{k\in\mathbb{Z}_N^2} U(k) J_k \,, \qquad U(k) = N\,\mathrm{tr}_{(N)}\left(U J_k^\dagger\right) . \tag{4.4}$$

We can interpret the momentum coefficients as the dynamical degrees of freedom in the TEK model.

In analogy with the continuum counterpart (2.3) we can define the operator (this time on a discrete torus)

$$\Delta(x) = \frac{1}{N^2}\sum_{k\in\mathbb{Z}_N^2} J_k\; e^{-2\pi i k_i x^i/L} \,, \tag{4.5}$$

where $L = aN$ is the dimensionful extension of the lattice with N^2 sites x^i. Because of the relations (4.2) the $\Delta(x)$ matrices are hermitean and periodic in x^i with period L, and thus the lattice is a discrete torus.

In analogy with the continuum formalism depicted in section 2, we can define an invertible map between $N\times N$ matrices and lattice fields. Namely, we have the following properties

$$\begin{aligned} \mathrm{tr}_{(N)}\left(J_k\Delta(x)\right) &= \frac{1}{N}e^{2\pi i k_i x^i/L} \,, \\ \frac{1}{N}\sum_x \Delta(x)_{\mu\nu}\Delta(x)_{\lambda\rho} &= \delta_{\mu\rho}\delta_{\nu\lambda} \,, \\ \frac{1}{N}\,\mathrm{tr}_{(N)}\left(\Delta(x)\Delta(y)\right) &= N^2\delta_{x,y \pmod L} \,. \end{aligned} \tag{4.6}$$

which yield a natural definition for the lattice field $\mathcal{U}(x)$ from the Fourier modes of its matrix partner U:

$$\mathcal{U}(x) \equiv \frac{1}{N}\sum_{k\in\mathbb{Z}_N^2} U(k) e^{2\pi i k_i x^i/L} = \frac{1}{N}\,\mathrm{tr}_{(N)}\left(U\Delta(x)\right) . \tag{4.7}$$

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Since

$$U = \frac{1}{N^2} \sum_x \mathcal{U}(x) \Delta(x) \,, \tag{4.8}$$

the unitarity condition on the matrix U is translated on the field $\mathcal{U}(x)$ in terms of U(1) *star unitarity*:

$$\mathcal{U}(x) \star \mathcal{U}^*(x) = \mathcal{U}^*(x) \star \mathcal{U}(x) = 1 \,, \tag{4.9}$$

where the *lattice star product* is defined by the natural discrete analog of eq. (2.10), namely

$$\begin{aligned} \mathcal{A} \star \mathcal{B} &\equiv \frac{1}{N} \operatorname{tr}_{(N)} \left(A B \Delta(x) \right) \\ &= \frac{1}{N^2} \sum_y \sum_z \mathcal{A}(x+y) \mathcal{B}(x+z) e^{2i(\theta^{-1})_{ij} y^i z^i} \,, \end{aligned} \tag{4.10}$$

with dimensionful non-commutativity parameter

$$\theta_{\mu\nu} = \frac{a^2 N}{\pi} n_{\mu\nu} \,. \tag{4.11}$$

The star product (4.10) reproduces the continuum version of eq. (2.10) in the limit $a \to 0$, and it reproduces the same algebraic properties with space-time integrals replaced by lattice sums.

In order to reproduce the non-commutative U(1) theory, we substitute the completeness relation

$$\frac{1}{N^2} \sum_x \Delta(x) = \mathbb{I}_N \tag{4.12}$$

into the action (3.7) and obtain

$$S_{\text{TEK}} = -\frac{\beta}{N} \sum_x \sum_\mu \operatorname{tr}_{(N)} \left[\left(U \Gamma_\mu U^\dagger \Gamma_\mu^\dagger \; + \text{h.c.} \right) \Delta(x) \right] . \tag{4.13}$$

As in the context of twisted reduced models, the matrices Γ_μ act as lattice shift operator, and thus they behave as discrete derivatives $e^{a \hat{\partial}_\mu}$. Indeed one can show from above that

$$\Gamma_\mu \Delta(x) \Gamma_\mu^\dagger = \Delta(x - a\hat{\mu}) \,, \tag{4.14}$$

from which it follows that shifts on the fields are represented as

$$\mathcal{U}(x + a\hat{\mu}) = \frac{1}{N} \operatorname{tr}_{(N)} \left(\Gamma_\mu U \Gamma_\mu^\dagger \Delta(x) \right) . \tag{4.15}$$

Therefore, we can rewrite the action (3.7) as

$$S_{\text{TEK}} = -\beta \sum_x \sum_\mu \left[\mathcal{U}(x) \star \mathcal{U}^*(x + a\hat{\mu}) + \mathcal{U}(x + a\hat{\mu}) \star \mathcal{U}^*(x) \right] . \tag{4.16}$$

4.1 The non-commutative theory

The theory described in eq. (4.16) is naturally U(1) left and right invariant, i.e., given a constant field $g \in$ U(1) the action is invariant under the transformations (where the ordinary product is intended)

$$\mathcal{U}(x) \to g \cdot \mathcal{U}(x)\,, \qquad \mathcal{U}(x) \to \mathcal{U}(x) \cdot g\,. \tag{4.17}$$

Let us turn to the commutative continuum version of the theory, whose field we call $u(x)$. First of all we perform the following substitution, dictated by the U(1) unitarity condition

$$u(x)u^*(x) = 1 \;\to\; u(x) = e^{i\varphi(x)}\,, \qquad \varphi \in \mathbb{R}\,. \tag{4.18}$$

The action then reads, up to the coupling constant

$$S = \int \mathrm{d}^2x \partial_\mu \left[e^{i\varphi(x)}\right] \partial^\mu \left[e^{-i\varphi(x)}\right] = \int \mathrm{d}^2x \partial_\mu \varphi(x) \partial^\mu \varphi(x)\,, \tag{4.19}$$

and therefore the theory is equivalent to the theory of a free massless real field.

Turning now to the non-commutative continuum theory of eq. (2.18), we notice that if the field $\mathcal{U}(x)$ decreased sufficiently rapidly at infinity we could integrate by parts and turn the star product into the standard one. The *action* would thus be the same as in the commutative version. Naturally, the field is subject to the constraint of star U(1) unitarity, which reads

$$\begin{aligned} \mathcal{U}(x) \star \mathcal{U}^*(x) &= 1 \\ &= \mathcal{U}(x)\mathcal{U}^*(x) + \sum_{n=1}^{\infty} \left(\frac{i}{2}\right)^n \frac{1}{n!} \theta^{i_1 j_1} \cdots \theta^{i_n j_n} \partial_{i_1} \cdots \partial_{i_n} \mathcal{U}(x) \partial_{j_1} \cdots \partial_{j_n} \mathcal{U}^*(x)\,. \end{aligned} \tag{4.20}$$

This condition naturally implies that the theory is not a free theory as in the commutative case, although the action would be formally the same. Moreover it is not clear if a scalar complex field subject to the condition (4.20) can at the same time satisfy the rapidly decreasing condition needed to integrate the non-commutative action by parts and neglecting the boundary behavior. Therefore, we will still use as the reference action of the continuum theory we wish to study eq. (2.18).

4.2 Physical interpretation: the brane vacuum

The low energy effective action for a p-brane in the presence of nonzero constant $B_{\mu\nu}$ field along the brane is given by the dimensional reduction of the 10-dimensional non-commutative Yang-Mills model to the brane world volume [10]. In particular, in the limit of large field $B_{\mu\nu}$, i.e. when

$$\alpha' \|B_{\mu\nu}\| \gg \|g_{\mu\nu}\|\,, \tag{4.21}$$

the non-commutativity parameter $\theta_{\mu\nu}$ is given by

$$\theta^{\mu\nu} = (B^{-1})^{\mu\nu}\,. \tag{4.22}$$

In the case of a brane-antibrane pair of a non-BPS non-stable brane, one finds in the spectrum of the effective theory also tachyonic modes, described by non-commutative scalar fields T with tachyonic potential $V(T)$ [11].

In the trivial non-commutative gauge field background, the part of the action of the brane system containing the tachyonic mode is given by the same action describing a non-commutative Higgs-like model of charged scalar fields, i.e.:

$$S = \int \mathrm{d}^{p+1}x \left(\frac{1}{2}\partial_\mu \phi \star \partial^\mu \phi^\dagger - V(\phi \star \phi^\dagger) \right), \tag{4.23}$$

where $V(\cdot)$ is a potential with a nontrivial v.e.v.: $|\phi|^2 =$ some constant, and the field ϕ transform in the bi-fundamental representation of the U(1) gauge group [12].

A point in the (true) vacuum of the field ϕ can be parametrized by an element of the non-commutative U(1),

$$\begin{aligned} \phi &\to U \star \phi \\ \phi^\dagger &\to \phi^\dagger \star U^{-1}, \end{aligned} \tag{4.24}$$

where U and U^{-1} satisfy $U \star U^{-1} = U^{-1} \star U = 1$.

To get the action describing the dynamics along the valley of the potential V in terms of the field U, one has to take ϕ constant at the minimum of the potential, perform the transformation (4.24) and declare U dynamical [7]. The action for the Goldstone *U-field* is then

$$S = \frac{1}{\lambda^2} \int \mathrm{d}^{p+1} \eta^{\mu\nu} \partial_\mu U \star \partial_\nu U^{-1}, \tag{4.25}$$

where $1/\lambda^2 = \phi \star \phi^\dagger$.

Our model then corresponds to the case $p = 1$, with euclidean metrics, and with the obvious identifications for the couplings λ, β and for the constant field $B_{\mu\nu}$ and the non-commutativity parameter of eq. (4.11) according to (4.22).

4.3 Double scaling limit

From (4.11) we see that in order to take the continuum limit of the model in such a way that the dimensionful non-commutativity parameter θ (which in the present case is just a real number) is fixed, we must fix the quantity $a^2 N$. It is clear from above that we have to send N to infinity if we want a to go to zero and the dimension of the lattice to go to infinity. In order to set $a \to 0$ we have to tune somehow the coupling constant β. From renormalization group analysis of the beta function of chiral models it is known that

$$a \sim \Lambda^{-1} e^{-c\beta}, \qquad c = 8\pi. \tag{4.26}$$

It is questionable whether a relation like (4.26) is valid in the context of the TEK reduced model, because whether the equivalence between principal chiral models and TEK reduced models holds is in itself a nontrivial question [9]. Moreover equation (4.26) is strictly valid only in the planar limit. Nevertheless, we would like to propose to *assume* such a relation, and to numerically verify its consistency.

An analogous assumption is made in [13], where the relation between a and β is taken to be the known Gross-Witten planar result. The numerical results presented in [13], incidentally, strongly confirm the validity of such hypothesis.

Whether a different value for c from the one indicated in (4.26) or a different functional dependence would lead to similar results to what we present in section 5 is a non trivial legitimate question.

What we propose to do is therefore to send N and β to infinity in such a way that $\vartheta \equiv N \cdot e^{-16\pi\beta}$ is kept fixed. Numerical analysis indicates that indeed finite N and β effects tend to compensate in such a limit, in a manner similar to the one obtained in [13] for the $2D$ EK model.

4.4 Correlation functions

Typical objects that can be studied in numerical simulations are correlations functions. In particular, the easiest one to compute turns naturally out to be the two points function. In particular, given a lattice site $n = (n_1, n_2)$, we define, in the reduced model, the following function, which is nothing but the translated version of

$$G(\mathbf{n}) \equiv \frac{1}{N} \left\langle \mathrm{Re}\,\mathrm{tr}_{(N)} \left(U(\mathbf{n}) U(0)^\dagger \right) \right\rangle \tag{4.27}$$

via the substitution of eq. (3.3), namely

$$G_{\mathrm{TEK}}(\mathbf{n}) \equiv \frac{1}{N} \left\langle \mathrm{Re}\,\mathrm{tr}_{(N)} \left(\Gamma_1^{n_1} \Gamma_2^{n_2} U (\Gamma_2^\dagger)^{n_2} (\Gamma_1^\dagger)^{n_1} U^\dagger \right) \right\rangle . \tag{4.28}$$

Again substituting the completeness relation (4.12) we get

$$G_{\mathrm{TEK}}(\mathbf{n}) = \frac{1}{N^2} \sum_x \langle \mathrm{Re}\,(\mathcal{U}(x + \mathbf{n}) \star \mathcal{U}^*(x)) \rangle . \tag{4.29}$$

Since the non-commutative theory is defined on a lattice with N^2 sites, this expression defines the average of the two point function over all the possible lattice sites, and thus gives a coherent expression for the two point function of the theory.

Incidentally, the internal energy of the model is given, up to constants, by $G_{\mathrm{TEK}}(1,0) + G_{\mathrm{TEK}}(0,1)$.

5. Numerical results

What should we expect from numerical Monte Carlo analysis? First of all, if the double scaling limit we take is correct, we should expect a coherent superposition of the behavior of the correlation functions as $N, \beta \to \infty$. If this is the case, it would strongly indicate the correctness of the procedure. Secondly what we should not expect is the typical behavior of the finite N corrections found in the reduced model in the strong coupling regime [9]. In the double scaling limit one takes into account also the non-planar diagrams, and it is therefore not clear how the finite N and β effects could manifest.

We used a Metropolis algorithm to update the $N \times N$ matrix U. Trial matrices were selected by multiplying the actual matrix U by a random SU(2) matrix embedded in

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	N	β
$\vartheta_1 = 9.93 \cdot 10^{-6}$	60	0.31063
	70	0.31370
	80	0.31635
	100	0.32079
	120	0.32442
$\vartheta_2 = 1.14 \cdot 10^{-6}$	50	0.35000
	80	0.35935
	100	0.36379
	120	0.36742

Table 1: Set of values of (N, β) at fixed ϑ which we investigated numerically.

SU(N), choosing randomly among the $N(N-1)/2$ SU(2) subgroups, and a U(1) random phase. More precisely, once the SU(2) subgroup and the U(1) phase were randomly chosen, we performed ten Metropolis hits with an approximate acceptance of 50%. Each SU(2) updating requires $O(N)$ operations. The number of SU(2)-subgroup updatings per run was $O(10^7)$. We should also mention that for the values of N we investigated (see table 1), we observed some problem of thermalization when using completely random configurations (for example, constructed by multiplying $N(N-1)/2$ completely random SU(2) matrix embedded in a $N \times N$ and associated to different subgroups, times a phase) as starting point of our simulations. In particular, we noted a worsening with increasing β. So, as starting point we used either moderately random matrices or the unity matrix. It is well known that a simple Metropolis algorithm does not provide a particularly efficient method to simulate a statistical system. Our choice was essentially due to the fact that the reduced action is quadratic in the matrix variable U. Moreover, it does not lend itself to a linearization by introducing new matrix variables, as in the case of the reduced TEK gauge theory [14].

We numerically studied the theory for two particular values of ϑ. Our choice was motivated by two facts. First of all it seems unnecessary to test the existence of a double scaling limit in the strong coupling region (i.e. for $\beta \lesssim \beta_c \approx 0.3058$) since the reduced theory is actually under control in that regime, and it has been shown [9] to accurately reproduce the standard, commutative parent theory. Secondly, we must address to sufficiently high values of N, because on the one hand the limiting procedure is expected to be sensible in the large N limit, and on the other hand the dual lattice of the reduced theory has dimensions proportional to N. Therefore, in order to avoid what we can legitimately call *finite size effects*, we had to resort to high values of N (the correlation length of the non reduced model is of the order of some lattice spacing in the region we investigated).

We choose to take the values reported in table 1, and to take $N \gtrsim 50$. A reasonable statistics for our Metropolis algorithm limited the highest N value to $N \lesssim 120$.

The correlation functions we studied was defined on the lattice as

$$\mathcal{G}(n) \equiv \frac{G_{\mathrm{TEK}}(n,0) + G_{\mathrm{TEK}}(0,n)}{2}\,. \tag{5.1}$$

In figure 1 and 2 we show our numerical results.

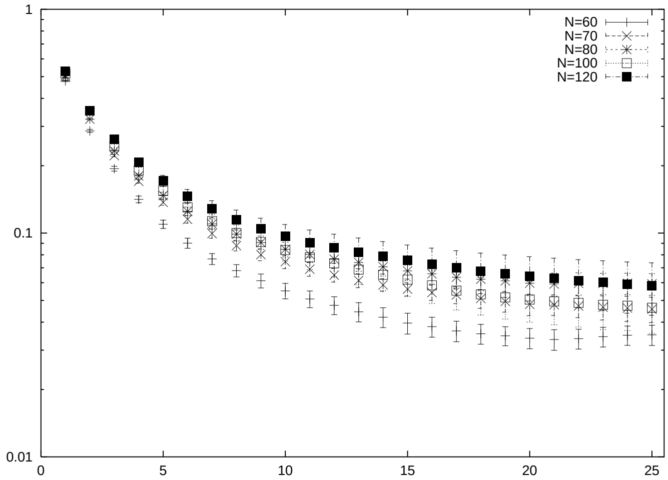

Figure 1: ϑ_1: Numerical results for $\mathcal{G}(n)$.

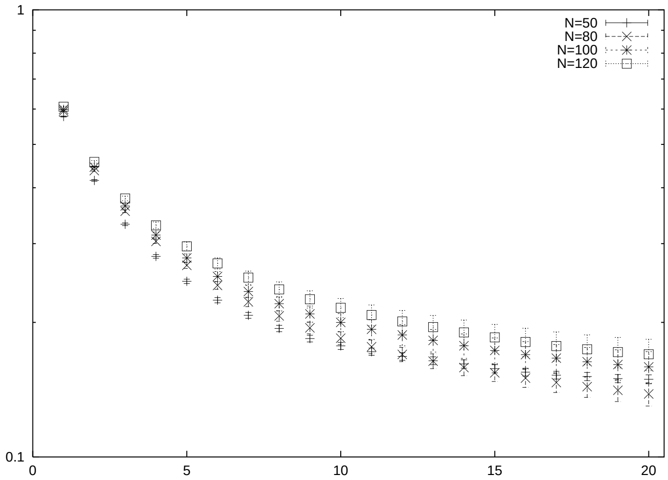

Figure 2: ϑ_2: Numerical results for $\mathcal{G}(n)$.

For large n, we see that asymptotically the obtained values for $\mathcal{G}(n)$ agree within error-bars. We should mention that we numerically studied also the so called *diagonal* two points correlation function on the lattice, defined by

$$\mathcal{G}_d(\sqrt{2} \cdot n) \equiv G_{\mathrm{TEK}}(n, n) \,. \tag{5.2}$$

The results we obtained for $\mathcal{G}_d(n)$ show an analogous asymptotic superposition within error-bars for large n in the double scaling limit, with values coherent with what was found for $\mathcal{G}(n)$. In fact, the lattice definition of the two point correlation function should not depend on either of the two definition one takes, and this can be viewed as another confirmation of the validity of the outlined scheme.

In all cases, what we find supports the expected numerical scenario, and confirms that the procedure described above indeed yields a sensible nonperturbative definition of the non-commutative theory we described.

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